Two Probabilistic Powerdomains in Topological Domain Theory

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Abstract

We present two probabilistic powerdomain constructions in topological domain theory. The first is given by a free "convex space" construction, fitting into the theory of modelling computational effects via free algebras for equational theories, as proposed by Plotkin and Power. The second is given by an observationally induced approach, following Schröder and Simpson. We show the two constructions coincide when restricted to ω-continuous dcppos, in which case they yield the space of (continuous) probability valuations equipped with the Scott topology. Thus either construction generalises the classical domain-theoretic probabilistic powerdomain. On more general spaces, the constructions differ, and the second seems preferable. Indeed, for countably-based spaces, we characterise the observationally induced powerdomain as the space of probability valuations with weak topology. However, we show that such a characterisation does not extend to non countably-based spaces.

1 Introduction

A well known problem in domain theory is to model probabilistic choice in a cartesian closed category of domains supporting an associated computability theory. Although the category of ω-continuous dcpos, which is the largest category typically considered for computability purposes, is closed under the probabilistic powerdomain construction, it is not known whether this holds for any of its cartesian closed subcategories [19].

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In a recent research project, see [7] for an overview, the authors have proposed to broaden the remit of domain theory by enlarging the collection of topological spaces admitted as domains to include spaces that do not carry Scott topologies. Such generalised domains are called *topological domains*, and they form a cartesian closed category that properly extends the category of ω-continuous dcpos. Although topological domains need not be countably based, they nonetheless admit *admissible representations* in the sense of the type two theory of effectivity [29, 23, 22], and thus support an associated treatment of computability. In this paper we show that topological domains are also closed under two natural probabilistic powerdomain constructions.

The first of our constructions is obtained by forming the free "convex space" in the category of topological domains. Such an approach results from a natural reworking, within topological domain theory, of the *abstract probabilistic domains* of Graham [11] and Jones and Plotkin [17, 18]. Nowadays, one appreciates this approach as fitting in nicely with Plotkin and Power's general programme of modelling computational effects via free algebras [21].

Our main results about the free convex space construction are as follows. First, such a construction exists in the category of topological domains. This is, in fact, an immediate consequence of a general existence results of Battenfeld for a wide range of computational effects [4]. Second, when restricted to ω-continuous dcppos, the free convex space construction coincides with the classical domain-theoretic probabilistic powerdomain of continuous probability valuations, cf. [17, 18]. Thus, we have a probabilistic powerdomain for topological domain theory that extends the classical domain-theoretic construction.

One might think that one should already be happy with this situation. However, one of the benefits of topological domain theory is that other natural spaces for modelling computation are included alongside the traditional domains with their Scott topologies. We observe that the free convex space construction is not particularly well behaved on such more general spaces.

To address this issue, we consider an alternative general approach to modelling probabilistic computation. Following Schröder and Simpson [25], we view a probabilistic powerdomain as being determined by the observations one can perform on probabilistic computations. This gives rise to a universal property that characterises an "observationally-induced" probabilistic powerdomain. That topological domains support a powerdomain enjoying such a universal property is then proved using a known connection between topological domain theory and realizability models [27, 3]. By this approach, we obtain a probabilistic powerdomain whose universal property we understand, but whose construction is given only abstractly and indirectly. The remaining effort of the paper is devoted to understanding this construction more explicitly.

On the positive side, we show that, for any countably based topological space X, the observationally-induced powerdomain over X is given by the space $\mathcal{V}_1(X)$ of continuous probability valuations with weak topology. Since it is known that the weak and Scott topologies coincide for valuations over continuous dcppos, cf. [14], when restricted to such spaces, the observationally-induced powerdomain again agrees with the classical domain-theoretic powerdomain, and hence with the first powerdomain construction discussed above. However, the new construction is better behaved on other collections of countably based spaces. On the negative side, we adapt an example in [9], to show that, for non countably based X, the observationally-induced powerdomain is not, in general, given by the space $\mathcal{V}_1(X)$.

2 Preliminaries

Domain theory and topological domain theory We write *dcpo* to mean a directed-complete partial order, and *dcppo* for a *pointed* dcpo, i.e. one with least element. For other domain-theoretic notation and terminology, the reader is referred to [1, 12].

For general topological concepts see, e.g., [28]. A topological space X is said to be a *qcb space* (*q*uotient of a *c*ountably *b*ased space) if it arises as a topological quotient $q\colon A \to X$ where A is a countably-based space. Topological domain theory derives from the (surprising) fact that the category **QCB** of continuous functions between qcb spaces is cartesian closed [20, 22, 10]. It is based on identifying an appropriatel notion of domain-like space within the collection of qcb spaces.

Recall that any topological space, X, carries a *specialization preorder* $\sqsubseteq$ defined by $x \sqsubseteq y$ iff every neighbourhood of x contains y. A space X satisfies the T_0 separation property if and only if $\sqsubseteq$ is a partial order. A qcb space, X, is said to be a *topological predomain* if: (i) it is T_0, and (ii) every ascending chain $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ in the specialization order has an upper bound x_∞ such that the sequence (x_i) converges to x_∞ in the topology. A *topological domain* is simply a topological predomain that has a least element in the specialization order. The categories **TP** of topological predomains and **TD** topological domains play respective roles in topological domain theory to those played by the categories of dcpos and dcppos in classical domain theory. Importantly, every ω-continuous dcpo is a topological predomain under its Scott topology, as, more generally, is every dcpo whose Scott topology is countably based. Furthermore, a continuous dcpo is a topological predomain under its Scott topology only if it is countably based. For further motivation, discussion and technical details, see the survey article [7].

Probabilistic powerdomains Let $\mathbb{I}^{\uparrow}$ denote $[0,1]$ eqipped with the Scott topology relative to the usual order. For a topological space X, let $\mathcal{O}(X)$ denote the lattice of open subsets of X equipped with the Scott topology.

A *(continuous)*[1] *subprobability valuation* over X is a continuous map $\nu : \mathcal{O}(X) \to \mathbb{I}^{\uparrow}$ such that $\nu(\emptyset) = 0$, and $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$. If additionally $\nu(X) = 1$, we call ν a *probability valuation* over X. There is a pointwise order on the set of (sub)probability valuations, given by $\nu \leq \nu'$, if for all $U \in \mathcal{O}(X)$, it holds that $\nu(U) \leq \nu'(U)$, making it a dcpo. By $\mathcal{V}^{\uparrow}_{\leq 1}(X)$, respectively $\mathcal{V}^{\uparrow}_{1}(X)$, we denote the space of subprobability valuations, respectively probability valuations, over X equipped with the Scott topology with respect to this ordering.

Jones and Plotkin use $\mathcal{V}^{\uparrow}_{\leq 1}(D)$, for a dcpo D, as their *probabilistic powerdomain* [18, 17]. This definition combines both probabilistic choice and nontermination into a single construction. Since it is both mathematically and computationally natural to separate the two, we shall revise their terminology. We call $\mathcal{V}^{\uparrow}_{1}(D)$ the *probabilistic powerdomain* and $\mathcal{V}^{\uparrow}_{\leq 1}(D)$ the *subprobabilistic powerdomain.* We consider the former to be the more basic construction since $\mathcal{V}^{\uparrow}_{\leq 1}(D)$ can be defined from it as $\mathcal{V}^{\uparrow}_{1}(D_{\perp})$.

A *point valuation* over X is a map $\delta_x : \mathcal{O}(X) \to \mathbb{I}^{\uparrow}$, which assigns 1 to U if $x \in U$ and 0 otherwise, for some $x \in X$. A finite subconvex combination $\sum_{i=1}^{n} \lambda_i \delta_{x_i}$ of point valuations, i.e. $\sum_{i=1}^{n} \lambda_i \leq 1$, is called a *simple* subprobabilistic valuation. Clearly, simple valuations are continuous subprobability valuations. A simple valuation is a probability valuation iff it is given by a convex combination, i.e. $\sum_{i=1}^{n} \lambda_i = 1$. For a dcpo D, the map $\delta \colon x \mapsto \delta_x$ is a continuous function from D to $\mathcal{V}^{\uparrow}_{1}(D)$ (and to $\mathcal{V}^{\uparrow}_{\leq 1}(D)$).

3 Free convex spaces as powerdomains

Our first approach probabilistic powerdomains in topological domain theory is based on considering probabilistic choice as an algebraic operation, following the general approach of Plotkin and Power to computational effects [21]. In classical domain theory, such an approach was taken by Graham [11]

[1] We drop the qualification continuous, since we only consider continuous valuations.

and Jones and Plotkin [17, 18], who defined the relevant notion of algebra. According to them, an *abstract probabilistic domains* is a dcppo D together with a continuous operation $+ : \mathbb{I} \times D^2 \to D$ (here $\mathbb{I} = [0,1]$ with Euclidean topology, D^2 carries the Scott-topology and the product is given the product topology), satisfying the equations below, where we write $x +_\lambda y$ instead of $+(\lambda, x, y)$.

$$\begin{aligned} x +_\lambda x &= x \\ x +_1 y &= x \\ x +_\lambda y &= y +_{1-\lambda} x \\ (x +_\lambda y) +_\mu z &= x +_{\lambda\mu} (y +_{\frac{\mu(1-\lambda)}{1-\lambda\mu}} z) \qquad\qquad (\lambda\mu \neq 1) \end{aligned}$$

For any continuous dcpo D, the evident weighted sum of subprobability valuations gives a canonical abstract probabilistic domain structure on the dcpo $\mathcal{V}^{\uparrow}_{\leq 1}(D)$ [17]. When D is a dcppo, this structure restricts to $\mathcal{V}^{\uparrow}_1(D)$.

A continuous function $f \colon A \to B$ between abstract probabilistic domains is said to be *affine* if it preserves the weighted sum structure, i.e. $f(x +_\lambda y) = f(x) +_\lambda f(y)$. In her thesis [17, Theorem 5.9], Claire Jones gave a characterisation of the subprobabilistic powerdomain $\mathcal{V}^{\uparrow}_{\leq 1}(D)$ over a continuous dcpo D as a free abstract probabilistic domain over D in a suitable sense. A mild, and more natural, variation of this result characterises the probabilistic powerdomain $\mathcal{V}^{\uparrow}_1(D)$.

Proposition 3.1. *For any continious dcppo D, abstract probabilistic domain A and continuous function $f \colon D \to A$, there exists a unique affine map $h \colon \mathcal{V}^{\uparrow}_1(D) \to A$ such that $f = h \circ \delta$. If f is strict then so is h.*

This proposition is interderivable with Theorem 5.9 of [17]. We do not give details since our use of the result is purely motivational.

The proposition above suggests, in general, defining a probabilistic powerdomain in topological domain theory as a free convex topological domain in the sense of the definition below.

Definition 3.2. A *convex qcb space* is a qcb space A supporting a continuous operation $+ : \mathbb{I} \times A^2 \to A$, where products are taken in **QCB**, subject to the four equational axioms, given above. A *convex topological (pre)domain*, is a convex qcb-space that also is a topological (pre)domain.

In the case that the qcb space A is a dcpo with Scott topology, then the the qcb power A^2 carries the Scott topology [6, Prop. 5.1], and the qcb product $\mathbb{I} \times A^2$ carries the product topology, cf. [6, Prop. 3.3]. Thus the continuity requirement on $+$ above agrees with that in the definition of abstract probabilistic domain. Once again, *affine maps*, as defined above, are the natural notion of homomorphism between convex qcb spaces.

The equational theory presented above fits into a general framework of parametrized equational algebraic theories, which, for topological domain theory, have been studied extensively by Battenfeld [4, 5]. The existence of free convex spaces is an immediate consequencec of his results.

Proposition 3.3. *For any qcb-space X, the free convex qcb-space $F_{\mathrm{conv}}X$ over X exists. Likewise the free typological predomain $\mathcal{F}_{\mathrm{conv}}X$ over X exists.*

Proof. The first statement follows from Theorem 4.7 of [4], the second from Corollary 5.5. □

The general construction of free qcb algebras in [4] enjoys the property that the underlying set of the a free qcb algebra coincides with that of the free set-theoretic algebra. It follows that the underlying set of the free convex qcb-space $F_{\mathrm{conv}}X$ can be described as follows. Its elements o are formal convex combinations over X, i.e. functions $\lambda\colon X \to [0,1]$ that are non-zero on finitely many points and satisfy $\sum_{x\in X} \lambda_x = 1$. We write such an element mnemonically as $\sum_{x\in X} \lambda_x\, x$. The action of the weighted sup operation on such elements is obvious. As defined in [4], the topology on $F_{\mathrm{conv}}X$ is given as a quotient. It is possible to give other more direct accounts of this topology, but instead we turn our attention to the free convex topological predomain, which we take as our first probabilistic powerdomain in topological domain theory.

For a wide class of equational theories, free algebras in topological predomains are obtained as the predomain reflection of the corresponding free qcb-algebras [4]. In particular, $\mathcal{F}_{\mathrm{conv}}(X)$ is constructed as $\mathfrak{M}(F_{\mathrm{conv(X)}})$ where $\mathfrak{M}$ is the left adjoint to the inclusion of topological predomains in **QCB**, as described in [6, 24]. Obtaining a more explicit description of $\mathcal{F}_{\mathrm{conv}}(X)$ in general seems tricky. Nevertheless, as our main result about $\mathcal{F}_{\mathrm{conv}}$, we obtain that it coincides with the classical probabilistic powerdomain when applied to ω-continuous dcppos.

Theorem 3.4. *For any ω-continuous dcppo D, we have that $\mathcal{F}_{\mathrm{conv}}D$ and $\mathcal{V}_1^{\uparrow}(D)$ are isomorphic by way of affine maps.*

The proof relies on a previous result of Battenfeld [5], which characterises the classical subprobabilistic powerdomain over an ω-continuous dcpo as a free algebra in topological predomains with respect to a different equational theory (see Section 4). This previous result thus already establishes a subprobabilistic powerdomain on the categiry **TP** which coincides with its domain-theoretic counterpart on ω-continuous dcpos. Here we are improving upon this result in two ways. First, we are considering the more natural (we believe) probabistic powerdomain (as we call it).[2] Second, the algebraic theory of convex spaces is more natural than the theory used in [5]. That these differences are not minor points is indicated by extra difficulties that arise in the proof of Theorem 3.4 above. We postpone the detaile to Section 4 below.

We end this section with various observations concerning $\mathcal{F}_{\mathrm{conv}}$. First, we observe that Theorem 3.4 does not extend to non-pointed continuous dcpos. Indeed, let $\mathbf{2}$ be the discrete two-point space. From the construction of the free convex qcb-space, one straightforwardly shows that $F_{\mathrm{conv}}\mathbf{2}$ is homeomorphic to $\mathbb{I}$. Hence, as a qcb space with discrete specialization order,, $F_{\mathrm{conv}}\mathbf{2}$ is a topological predomain, and thus it is the free convex topological predomain $\mathcal{F}_{\mathrm{conv}}\mathbf{2}$ over $\mathbf{2}$. However, it does not carry the Scott topology, which is discrete, and hence differs from $\mathcal{V}_1^{\uparrow}(\mathbf{2})$. Of course, the Euclidean topology is the natural topology in this case. Indeed, the discrete topology is not even a qcb space. The moral of this example is that topological predomains are a better environment than dcpos for modelling (total) probabilistic computation over discrete data.[3]

Using Theorem 3.4, one easily seess that, in contrast to the previous example, the constructions $F_{\mathrm{conv}}X$ and $\mathcal{F}_{\mathrm{conv}}X$ do not coincide in general. For example consider the space $\overline{\omega} =_{\mathrm{def}} \mathbb{N} \cup \{\infty\}$, with the usual order and Scott topology. The probability valuations on $\overline{\omega}$ are in one-to-one correspondence with *countable* convex combinations of elements. Thus, for example, there is a probability valuation corresponding to the weighted sum $\sum_{i \in \mathbb{N}} 2^{-(i+1)}\, i$, which is thus an element of $\mathcal{F}_{\mathrm{conv}}(\overline{\omega})$ but not of $F_{\mathrm{conv}}(\overline{\omega})$.

Generalising the above, it is known that probability valuations on a continuous dcpo D are in one-to-one correspodence with Radon probability measures on D. Thus Theorem 3.4 may tempt one to view $\mathcal{F}_{\mathrm{conv}}X$ as a generalised space of probability measures, over a qcb space X, of a form that is compatible with topological domain theory. However, such a view is misleading. For example the space $\mathcal{F}_{\mathrm{conv}}(\mathbb{I})$ coincides with $F_{\mathrm{conv}}(\mathbb{I})$, since the latter can be shown to be Hausdorff (and hence is a topological predomain). So, in this case, $\mathcal{F}_{\mathrm{conv}}\mathbb{I}$ is merely a space of convex combinations, which barely scratches the surface of the collection of probability measures over $\mathbb{I}$.

[2] We repeat, our terminology differs from much of the literature.

[3] Arguably, such purely total spaces lie outside the remit of domain theory, which typically models total elements as limits of partial approximations. But such spaces of total elements do unquestionably lie within the scope of topology. Topological domain theory provides a natural environment that encompasses both worlds.

The last example raises the question of whether it is possible to find a construction of a probabilistic powerdomain on topological predomains that behaves better on discretely-ordered spacecs such as $\mathbb{I}$. We shall answer this question affirmatively in Section 5.

4 Proof of Theorem 3.4

If D is a dcppo, we denote by $D^\dagger$ the dcpo $D\backslash\{\perp_D\}$. Notice that $(-)^\dagger$ is not functorial. It is a trivial observation that D is a continuous dcppo if and only if $D^\dagger$ is a continuous dcpo. Also, the probabilistic powerdomain $\mathcal{V}_1^\uparrow(D)$ is isomorphic to the subprobabilistic powerdomain $\mathcal{V}_{\leq 1}^\uparrow(D^\dagger)$.

In [5], Battenfeld has shown that the subprobabilistic powerdomain over an ω-continuous dcpo can be recovered in **TP** as a free "interval cone" $F_{\mathrm{IM}}D^\dagger$ over $D^\dagger$, which will be defined below. Probabilistic domains in classical domain theory have been described as dcpo-cones in the literature, see e.g. [12], the equational theory for interval cones given below is taken from Heckmann [13], who compared several algebraic approaches to probabilistic domains. We will use his results to show that $F_{\mathrm{conv}}D \cong F_{\mathrm{IM}}D^\dagger$.

The reason for using the theory of interval cones is that the methods used in [5] to derive the coincidence of various free algebra constructions in topological and classical domain theory are only applicable to parametrized equational theories in which the parameter spaces carry the Scott topology. This is not the case for the convex spaces of Section 3, since there the parameter space is $\mathbb{I}$. In contrast, interval cones do satisfy this condition.

Definition 4.1. A *qcb-interval cone* is a qcb-space A with a constant $\perp$, and continuous operations $\oplus : A^2 \to A$ and $\cdot : \mathbb{I}^\uparrow \times A \to A$, subject to the following axioms:

$$\begin{array}{cc}
0 \cdot x = \perp & x \oplus x = x \\
1 \cdot x = x & x \oplus y = y \oplus x \\
\lambda \cdot \perp = \perp & (x \oplus y) \oplus (u \oplus v) = (x \oplus u) \oplus (y \oplus v) \\
\lambda \cdot (\mu \cdot x) = \lambda\mu \cdot x & \lambda \cdot (x \oplus y) = \lambda \cdot x \oplus \lambda \cdot y \ \ . \\
 & (\lambda + \mu)/2 \cdot x = \lambda \cdot x \oplus \mu \cdot x
\end{array}$$

A **TP**-*interval cone*, is a qcb-interval cone that also is a topological predomain. A continuous map $h : A \to B$ between qcb-interval cones is called an *IM-homomorphism*, if $h(\perp) = \perp$, for all $x, y \in A$, $h(x \oplus y) = h(x) \oplus h(y)$, and for all $\lambda \in \mathbb{I}^\uparrow$, $h(\lambda \cdot x) = \lambda \cdot h(x)$.

Using results of [4], we get again that free algebras for this equational theory exist in **QCB** and **TP**.

Proposition 4.2. *For any qcb-space X, the free qcb-interval cone $F_{\mathrm{IM}}X$ over X exists. Likewise the free* **TP***-interval cone.*$\mathcal{F}_{\mathrm{IM}}X$

Proposition 4.3. *For an ω-continuous dcpo D, $\mathcal{F}_{\mathrm{IM}}D$ is isomorphic to the subprobabilistic powerdomain $\mathcal{V}^{\dagger}_{\leq 1}D$..*

Proof. This is Theorem 4.17 of [5]. ☐

We now show that for a continuous dcpo D, the free **TP**-interval cone over D is isomorphic to the free convex topological predomain over $D_\perp$. For this, we need to two lemmas, whose proofs can be found in Appendix A.

Lemma 4.4. *Any* **TP***-interval cone A, that is a continuous dcpo, is a convex topological predomain. Moreover, any* IM*-homomorphism $h : A \to B$ into a* **TP***-interval cone that also is a convex topological predomain, is a strict affine map.*

Lemma 4.5. *Any convex qcb-space A whose specialization order has a least element is a qcb-interval cone. Moreover, any strict affine map $h : A \to B$ into a convex qcb-space with least element is an* IM*-homomorphism.*

Theorem 4.6. *For any ω-continuous dcpo D, it holds that $\mathcal{F}_{\mathrm{IM}}D$ is isomorphic to $\mathcal{F}_{\mathrm{conv}}D_\perp$. Equivalently, for any ω-continuous dcppo D, it holds that $\mathcal{F}_{\mathrm{conv}}D$ is isomorphic to $\mathcal{F}_{\mathrm{IM}}D^{\dagger}$.*

Proof. Let D be a continuous dcpo, $\eta_D : D \hookrightarrow \mathcal{F}_{\mathrm{IM}}D$ and $\eta^*_{D_\perp} : D_\perp \hookrightarrow \mathcal{F}_{\mathrm{conv}}D_\perp$ be the inclusion maps into the free algebras, and $\iota : D \hookrightarrow D_\perp$ the inclusion map, obtained from the lifting. Then by Proposition 4.3, $\mathcal{F}_{\mathrm{IM}}D$ is a continuous dcpo, hence by Lemma 4.4 a convex topological predomain. Moreover, $\mathcal{F}_{\mathrm{conv}}D_\perp$ is a convex qcb-space with a least element, namely $\eta^*_{D_\perp}(\perp)$. Thus, by Lemma 4.5, $\mathcal{F}_{\mathrm{conv}}D_\perp$ is a **TP**-interval cone. Observe that the distinguished element $\perp \in \mathcal{F}_{\mathrm{IM}}D$ is the least element w.r.t. the specialization order, as whenever $\perp = 0 \cdot x \in \mathcal{F}_{\mathrm{IM}}D$, then $x = 1 \cdot x \in U$, by continuity of $\cdot$. Thus $\mathcal{F}_{\mathrm{IM}}$ is a topological domain, and η_D factors through ι, as $\eta_D = \alpha \circ \iota$, and α is strict. Now the universal properties of free algebras yield the following two commuting diagrams:

$$\begin{tikzcd} \mathcal{F}_{\mathrm{IM}}D \arrow[r, dashed, "h"] & \mathcal{F}_{\mathrm{conv}}D_\perp \\ D \arrow[u, "\eta_D"] \arrow[ur, "\eta^*_{D_\perp}\circ\iota"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}_{\mathrm{conv}}D_\perp \arrow[r, dashed, "h^*"] & \mathcal{F}_{\mathrm{IM}}D \\ D_\perp \arrow[u, "\eta^*_{D_\perp}"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

where h is the unique IM-homomorphism extending $\eta^*_{D_\perp} \circ \iota$, and h^* the unique affine map extending α, which is strict by α being strict. But, by Lemma 4.4, h is a strict affine map, and by Lemma 4.5, h^* an IM-homomorphism. Thus, the universal property yields that $h \circ h^* \cong \mathrm{id}_{\mathcal{F}_{\mathrm{conv}}D_\perp}$ and $h * \circ h \cong \mathrm{id}_{\mathcal{F}_{\mathrm{IM}}D}$, as required. □

Theorem 3.4 follows by combining Theorem 4.6 and Proposition 4.3.

5 An observationally-induced powerdomain

In this section we implement a different approach to defining a probabilistic powerdomain in topological domain theory, adapting the approach of [25] to qcb spaces. The idea is to implement a universal property for probabilistic powerdomains, based on endowing them with as much structure as is consistent with certain properties of observations on probabilistic computations.

Instead of the convex qcb spaces of the previous section, we consider more primitive structures. A *(qcb) choice algebra* $(A, \oplus)$ is simply a qcb space together with a continuous map $\oplus\colon A \times A \to A$. The idea is that A is a space of probabilistic computations, and that $x \oplus y$ corresponds to the computation that makes a fair probabilsitic choice (coin toss) and then, depending on the outcome, continues as computation x or as computation y. One naturally expects some equational properties to hold of $\oplus$. However, as we shall see below, we shall not need to take them as basic.

The probabilistic powerdomain over an object X should represent a sensible space of probabilistic computations that output values in X. At the very least, such a space A should carry choice algebra structure, and there should be a continuous function $\delta\colon X \to A$ that maps any $x \in X$ to an element $\delta_x \in A$ corresponding to the deterministic computation that returns x. A further and crucial constraint is the following. The only natural way of performing a (probabilistic) observation on a computation that probabilistically returns a value in X is to perform a probabilstic observation on the returned value, with the resulting probability distribution over all possible outcomes being aggregated on the basis of the probabilistic choices made during the computation. We translate this into the following definition. For more detailed motivation, see [25].

Definition 5.1. An *abstract choice structure over* X is given by a choice algebra $(A, \oplus)$ and a continuous function $\delta\colon X \to A$ subject to the following condition. For any qcb space Z and continuous $Z \times X \to \mathbb{I}^{\uparrow}$ there exists a unique continuous $h\colon Z \times A \to \mathbb{I}^{\uparrow}$ satisfying:

$$h(z,\, \mu \oplus \mu') \;=\; \frac{1}{2}\,(h(z,\mu) + h(z,\mu')) \tag{1}$$

$$h(z,\, \delta(x)) \;=\; f(z,x)\ . \tag{2}$$

The definition above differs from that in [25] by the inclusion of the parameter space Z. When the definition is made in the category of all topological spaces, as in [25], the parameter space is not needed since the parametric property follows from the non-parametric one. We do not know if this also holds for qcb spaces, hence the formulation above.

We shall require our probabilistic powerdomain over X to be a an abstract choice structure. But some further condition is needed since arbitrary abstract choice structures over X satisfy two deficiencies: they need not model all ways of probabilistically computing values in X, and they need not satisfy the expected equational properties of probabilistic choice. (For example, the absolutely free algebra on one binary operation over X gives an abstract choice structure.) Both deficiencies are addressed simultaneously by defining a notion of completeness, which is further motivated in [25].

Definition 5.2. A choice algebra $(B, \oplus')$ is said to be *complete* if, for any abstract choice structure $\delta\colon X \to (A, \oplus)$, qcb space Z and continuous $f\colon Z \times X \to B$, there exists a unique continuous $h\colon Z \times A \to B$ satisfying

$$\begin{aligned} h(z,\, \mu \oplus \mu') \;&=\; h(z,\mu) \oplus' h(z,\mu') \\ h(z,\, \delta(x)) \;&=\; f(z,x)\ . \end{aligned}$$

It is immediate from the definition that the choice algebra $(\mathbb{I}^{\uparrow}, \oplus)$, where $\lambda \oplus \lambda' =_{\mathrm{def}} \frac{1}{2}(\lambda + \lambda')$, is complete.

Definition 5.3. The *observationally-induced probabilistic powerdomain over* X, is given by an abstract choice structure $\delta\colon X \to (\mathcal{F}_{\mathrm{obs}}(X), \oplus)$ where the choice algebra $(\mathcal{F}_{\mathrm{obs}}(X), \oplus)$ is complete.

We remark that this definition characterises $(\mathcal{F}_{\mathrm{obs}}(X), \oplus)$ up to (homomorphic) isomorphism. Indeed it simultaneously imposes two universal properties on the abstract choice structure $\delta\colon X \to (\mathcal{F}_{\mathrm{obs}}(X), \oplus)$. On the one hand, this is characterised as giving the initial complete choice algebra over X. On the other, it is final amongst all abstract choice structures over X.

Theorem 5.4. *For any qcb space X, the observationally-induced probabilistic powerdomain $\mathcal{F}_{\mathrm{obs}}(X)$ exists.*

Proof. The proof makes essential use of the fact that the category of T_0-qcb spaces is equivalent to the category of *extensional pers* in the realizability topos over Scott's combinatory algebra $\mathcal{P}\omega$, a proof of which appears in [3]. It is known that, in any realizability topos with $\neg\neg$-separated dominance Σ, the category of extensional pers is equivalent to a full reflective exponential ideal in the topos that is weakly small in an appropriate internal sense, cf. [16]. We call the objects in this subcategory the *extensional objects*. Such objects are defined internally in the topos as double-negation-closed subobjects of powers of Σ. Working internally in the topos, we formulate the category of complete choice algebras and homomorphisms, using the definitions above (replacing qcb spaces with extensional objects). One can prove straightforwardly that the forgetful from this category to the category of extensional objects creates limits. Thus the category of complete choice algebras is complete. It is also inherits the property of being weakly small from the category of extensional objects. By an (internal) application of Freyd's adjoint functor theorem, one obtains a left ajoint to this forgetful constructing internal free complete choice algebras.

The required observationally-induced probabilistic powerdomain is now defined as follows. Given a qcb space X, let X' be its T_0 reflection, viewed as an extensional object. The probabilistic powerdomain is obtained by applying the internal free complete choice algebra construction to X' and then externalising the result.

That this procedure does indeed produce the free complete choice algebra in the external sense above is then merely a consequence of the category of complete choice algebras as defined internally being (externally) equivalent to the external category of qcb-based complete choice algebras. This is established by simply unwinding the internal definition. However, the argument does make essential use of the parametrized aspect of Definitions 5.1 and 5.2 above. □

The proof of Theorem 5.4 does not give much information about the construction of $\mathcal{F}_{\mathrm{obs}}(X)$. Nevertheless, one can exploit the universal property of $\mathcal{F}_{\mathrm{obs}}(X)$ to obtain useful properties. Here we give just one example. In contrast to the free convex space construction, there is no need to reflect the observationally-induced probabilistic powerdomain into the category of predomains. Rather, this is automatically ensured by the assumed completenes property.

Proposition 5.5. *For any complete choice algebra $(A, \oplus)$, the space A is a topological predomain. In particular, $\mathcal{F}_{\mathrm{obs}}(X)$ is a topological predomain.*

Proof. Let $\overline{\omega}$ be the space $\mathbb{N} \cup \{\infty\}$ with Scott topology. and let ω be $\overline{\omega} \backslash \{\infty\}$ with subspace topology. A qcb space A is a topological predomain if and only if, for every qcb space Z and continuous $f : Z \times \omega \to A$, there exists a unique continuous $\overline{f} : Z \times \overline{\omega} \to A$ that agrees with f on $Z \times \omega$. We show that this property holds for any complete choice algebra $(A, \oplus)$.

Consider the continuous function $\delta : \overline{\omega} \to \mathcal{F}_{\mathrm{obs}}(\overline{\omega})$. Using the above characterisation of topological predomains, the fact that $\mathbb{I}^{\uparrow}$ is a topological predomain, and that δ is an abstract choice structure, one shows easily that the restriction $\delta' : \omega \to \mathcal{F}_{\mathrm{obs}}(\overline{\omega})$ is also an abstract choice structure.

Now to show A is a predomain, consider any $f : Z \times \omega \to A$. Using that δ' is an abstract choice structure and the completeness of A, one obtains a continuous $h : Z \times \mathcal{F}_{\mathrm{obs}}(\overline{\omega}) \to A$ that is a homomophism in its second argument. By restricting this along δ, one obtains the desired $\overline{f} : Z \times \overline{\omega} \to A$, whose uniqueness property follows from that of h. □

The above proof easily extends to show that the underlying spaces of complete choice algebras are *replete* in the sense of [15], relative to Sierpinski space. Other examples of consequences of the universal property (definition of integration, equational properties of integration, a monotone convergence theorem, Fubini theorem) are presented in [25], in the case of the obsrvationally-induced probabilistic powerdomain in the category of topological spaces. Identical arguments apply to the present setting of qcb spaces.

The main result of [25] is an explicit characterisation of the observationally-induced probabilistic powerdomain for a general topological space X as the space $\mathcal{V}_1(X)$ of probability valuations equipped with the *weak topology*, for which a subbase is given by sets of the form

$$\langle U, r \rangle =_{\mathrm{def}} \{\nu \in \mathcal{V}_{\leq 1}(X) | \; \nu(U) > r\} \enspace ,$$

where $U \in \mathcal{O}(X)$ and $r \in [0, 1)$. The name "weak" reflects that this topology is the coarsest topology relative to which integration $\nu \mapsto \int f \mathrm{d}\mu$ is continuous for every continuous $f : X \to \mathbb{I}^{\uparrow}$.[4] For the qcb powerdomain, we obtain a similar characterisation for a wide collection of spaces.

[4] Some authors prefer *weak* topology*, in recognition of a known characterisation of $\mathcal{V}_1(X)$ as a space of functionals.

Theorem 5.6. *For any countably-based space X, the space $\mathcal{F}_{\mathrm{obs}}(X)$ is given by the space $\mathcal{V}_1(X)$ of probability valuations on X with weak topology.*

Before outlining the proof of the theorem, we state some consequences. First, the observationally-induced powerdomain again coincides with the classical domain-theoretic construction, when restricted to ω-continuous dcppos.

Corollary 5.7. *If D is an ω-continuous dcppo then $\mathcal{F}_{\mathrm{obs}}(X)$ is the space $\mathcal{V}_1^{\uparrow}(X)$ of probability valuations with Scott topology.*

This follows immediately from [14], where it is shown that the weak topology and Scott topology coincide for the space of subprobability valuations over a continuous dcpo.

Further, and addressing one of the identified weaknesses of the free convex space construction, standard spaces of probability measures from mathematical analysis are recovered for non-domain-like spaces.

Corollary 5.8. *If X is a compact Hausdorff qcb space then $\mathcal{F}_{\mathrm{obs}}(X)$ is isomorphic to the compact Hausdorff space of probability measures with weak (a.k.a. vague) topology.*

Proof. Compact Hausdorff spaces are locally compact, and locally compact qcb spaces are countably based. Thus X is countably based, and hence every probability measure is a Radon measure. The result then follows from [2], where the coincidence of the weak topology on valuations with vague topology on Radon measures is shown, more generally, for stably compact spaces. $\square$

We now turn to the proof of Theorem 5.6. As in [25], our main tool is the following result.

Proposition 5.9. *For topological spaces Z, X and continuous $f\colon Z \times X \to \mathbb{I}^{\uparrow}$ there exists a unique continuous $h\colon Z \times \mathcal{V}_1(X) \to \mathbb{I}^{\uparrow}$ satisfying equations (1) and (2) above. Explicitly, this is given by the function*

$$h(z, \mu) \;=\; \int (x \mapsto f(z, x))\, \mathrm{d}\mu \ ,$$

using the standard notion of integration w.r.t. valuations, cf. [17].

Note that the products in this proposition are given the product topology.

The proof of the above proposition is long and uses the Axiom of Choice. There are two main steps, both nontrivial. The first establishes a similar uniqueness result for continuous functions $h\colon Z \times \mathcal{V}(X) \to [0, \infty]^{\uparrow}$ that are linear in their second argument and extend f, where $\mathcal{V}(X)$ is the space of all

$[0, \infty]$-weighted valutions with weak topology. This step provides a positive solution to a problem posed by Heckmann [14, Problem 1].[5] The second step applies the first to derive the stated unique extension property out of $\mathcal{V}_1(X)$. The details will appear in [26].

Proof of Theorem 5.6. Let X be countably based. We first show that $\delta\colon X \to (\mathcal{V}_1(X), \oplus)$ is an abstract choice structure. Let Z be a qcb space, and consider any $f\colon Z \times X \to \mathbb{I}^{\uparrow}$. By exhibiting Z as a topological quotient $q\colon Y \to Z$, where Y is countably based, we obtain $g\colon Y \times X \to \mathbb{I}^{\uparrow}$, by $g(y, x) = f(q(y), x)$. Since Y and X are both countably based, the qcb product $Y \times X$ carries the product topology. Hence, by Proposition 5.9, there is a unique continuous $h\colon Y \times \mathcal{V}_1(X) \to \mathbb{I}^{\uparrow}$, where again the topological and qcb products agree, since the weak topology preserves countable bases. satisfying the stated properties. But $h(y, x) = \int (x \mapsto f(q(z), x))\, \mathrm{d}\mu$ and qcb products preserve topological quotients, so $(q \times 1)\colon Y \times \mathcal{V}_1(X) \to Z \times \mathcal{V}_1(X)$ is a quotient. It follows that the function $h'\colon Z \times \mathcal{V}_1(X) \to \mathbb{I}^{\uparrow}$ defined by $h'(q(y), x) = h(y, x)$ is continuous. This is the required right-homomorphic extension of f. Its uniqueness is a direct consequence of the uniqueness property of h.

It remains to show that the choice algebra $(\mathcal{V}_1(X), \oplus)$ is complete. Suppose $d\colon Y \to (B, \oplus')$ is an abstract choice structure, and $f\colon Z \times Y \to \mathcal{V}_1(X)$ is continuous. For any open $U \subseteq X$, the function $(z, y) \mapsto f(z, y)(U)\colon Z \times Y \to \mathbb{I}^{\uparrow}$ is continuous, and hence has a unique right-homomorphic extension $h_U\colon Z \times B \to \mathbb{I}^{\uparrow}$. Then the function mapping (z, b) to $\lambda U.\, h_U(z, b)$ is the required extension of f, for it is easily checked that its image consists of probability valuations, and it is continuous because the weak topology on $\mathcal{V}_1(X)$ is the subspace topology from the topological power $(\mathcal{I}^{\uparrow})^{|\mathcal{O}(X)|}$. $\square$

To end the paper, we show that, in contrast to the situation for the observationally-induced powerdomain for general topological spaces of [25], for the qcb version of the present paper, the characterisation of Theorem 5.6 does not extend to non-countably-based spaces.

The following space appears in a recent paper by Gruenhage and Streicher [9]. Let G have as underlying set $\mathbb{N} \times \mathbb{N}$ with the topology is given as follows. For $(n, m) \in G$ a basic open neighbourhood is of the form

$$U((n, m), f) = \{(n, m)\} \cup \{(k, l) \in G |\; k > n \,\text{and}\, l \geq f(k - n)\},$$

where $f : \mathbb{N} \to \mathbb{N}$ is any set-theoretic map. In [9] it is shown that G is a (nonsober) qcb-space, whose sobrification is not a qcb-space.

[5] We remark that a special case of this step, in which X is assumed to be compactly generated, has been established by different methods as the main result of [8].

Theorem 5.10. *The space $\mathcal{V}_1(G)$ is not a qcb space. Hence $\mathcal{F}_{\mathrm{obs}}(G)$ is not isomorphic to $\mathcal{V}_1(G)$.*

The proof is included in Appendix B.

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A Proofs of lemmas from Section 4

Lemma A.1. *Let X, Y be arbitrary topological spaces and D be a continuous dcpo. Then a map $f : X \times D \to Y$, where $X \times D$ carries the product topology, is continuous if and only if it is continuous in each argument separately.*

Proof. Trivially a jointly continuous map is continuous in each argument. For the converse, let $V \subseteq Y$ be open, and $f(x_0, d_0) \in V$. We show the existence of an open neighbourhood $W \subseteq f^{-1}(V)$ in the product topology of $X \times D$, which contains (x_0, d_0). By continuity in the second argument and the fact that D is a continuous dcpo, there exists $d_1 \ll d_0$ such that $f(x_0, d_1) \in V$. Next, by continuity in the first argument, we find an open neighbourhood $U \subseteq X$ of x_0 such that for all $x \in U$, $f(x, d_1) \in V$. Thus, for all $x \in U$ and $d \gg d_1$, $f(x, d) \in V$, hence setting $W = U \times \twoheaduparrow d_1$ yields the required open subset $W \subseteq f^{-1}(V)$. □

Proof of Lemma 4.4. A proof of the first part can be found in [13, §5–7]. We sketch the idea. Using the equations of IM, each term can be rewritten as a finite tree, such that all occurrences of $\cdot$ are at the leaves, and we assign the weight $\frac{\lambda}{2^k}$ to each leaf, where λ is the first argument of the $\cdot$-operation at the leaf, and k is the number of branches from the root to the leaf, e.g. for $(\lambda_1 \cdot x \oplus \lambda_2 \cdot y) \oplus \lambda_3 \cdot z$, we assign weight $\frac{\lambda_1}{4}$ to x, $\frac{\lambda_2}{4}$ to y, and $\frac{\lambda_3}{2}$ to z. Using the equations one can show that terms resulting in the same weights are indeed equivalent in A. Thus, we get elements of the form $\langle \frac{\lambda_1}{2^{k_1}}, x_1; \ldots; \frac{\lambda_n}{2^{k_n}}, x_n \rangle$, where $\sum_{i=1}^n \frac{1}{2^{k_i}} = 1$. Now for a dyadic rational $d \in \mathbb{D}$, define $x +_d y$ to be the term resulting in $\langle d, x; 1-d, y \rangle$, i.e. all the leaves are labelled x or y and all the λ_i equal 1. Next, we use the fact that topological predomains have least upper bounds and define $x +_\lambda y$ as the least upper bound of terms resulting in $\langle \frac{\lambda_i}{2^{k_i}}, x; \frac{\mu_i}{2^{l_i}}, y \rangle$ such that $\{\frac{\lambda_i}{2^{k_i}}\}_i$ is increasing with $\bigvee^{\uparrow} \frac{\lambda_i}{2^{k_i}} = \lambda$, and $\{\frac{\mu_i}{2^{l_i}}\}_i$ is increasing with $\bigvee^{\uparrow} \frac{\mu_i}{2^{l_i}} = 1 - \lambda$. This construction will result in a well-defined operation $+ : \mathbb{I} \times A^2 \to A$, which is continuous in each argument separately. Thus, Lemma A.1 and the fact that A is a continuous dcpo, guarantee that $+$ is jointly continuous, and hence A a convex topological predomain.

For the second part, strictness of h is immediate, as $\perp$ is the least element of the specialization order of a **TP**-interval cone. Now observe that, using the above idea, it is clear that $h(x +_d y) = h(x) +_d h(y)$ for all dyadic rationals d. However, the space of dyadic rationals $\mathbb{D}$ is a dense subspace of $\mathbb{I}$. So, using the fact that $\mathbb{I}$ is Hausdorff, the map $(d, x, y) \mapsto h(x +_d y)$ can be uniquely extended to all elements of the unit interval in the first component, showing that h is indeed affine. □

Proof of Lemma 4.5 Let $\perp$ be the least element of A, and define $x \oplus y$ as $x +_{\frac{1}{2}} y$ and $\lambda \cdot x$ as $x +_\lambda \perp$. Then all operations for an interval cone are well-defined, and $\oplus$ is obviously continuous. For the continuity of $\cdot$, observe that it is clearly continuous as a map $\mathbb{I} \times A \to A$, so we only have to show that for all open $V \subseteq A$, the first component of $\cdot^{-1}(V)$ is upper closed with respect to the usual order of $\mathbb{I}$. So let $\lambda \cdot x \in V$, and $\lambda < \mu$. Then $x +_\lambda \perp \in V$, hence $x +_\lambda (x +_{\frac{\mu-\lambda}{1-\lambda}} \perp) \in V$, as $\perp$ is the least element of A, and $+$ continuous. But the last term evaluates to $x +_\mu \perp$, showing that the first component of $\cdot^{-1}(V)$ is upper closed, indeed. Showing that all the IM-axioms hold in A is a straightforward task, and left to the reader.

For the second part, observe that all IM-operations are defined from the convex operation $+$. Thus a strict affine map, clearly is an IM-homomorphism, as required. $\square$

B Proofs of Theorem 5.10

Recall that the elements of the sobrification of a topological space X are the complete prime filters of $\mathcal{O}(X)$. For $\lambda \in \mathbb{I}^{\uparrow}$ and $\mathcal{F} \in Sob(X)$, we denote by $\lambda\chi_{\mathcal{F}} : \mathcal{O}(X) \to \mathbb{I}^{\uparrow}$ the continuous map, assigning λ to U if $U \in \mathcal{F}$, and 0 otherwise. Similarly, we define for $\sum_{\mathcal{F}\in Sob(X)} \lambda_{\mathcal{F}} \leq 1$, $\sum_{\mathcal{F}\in Sob(X)} \lambda_x \chi_{\mathcal{F}}$ to be the continuous map assigning to $U \in \mathcal{O}(G)$ the value $\sum_{U\in\mathcal{F}} \lambda_{\mathcal{F}}$. Under this construction, $Sob(X)$ becomes the topological subspace of $\mathcal{V}_1(X)$, which consists of valuations of the form $\chi_{\mathcal{F}}$, for $\mathcal{F} \in Sob(X)$ (the point valuations over $Sob(X)$).

The elements of the sobrification ofthe space G are exactly the neighbourhood filters $\mathcal{F}_x$ for $x \in G$, and $\mathcal{F}_\infty := \mathcal{O}(G)\setminus\{\emptyset\}$ (see [9]). Thus $Sob(G)$ becomes the subspace of $\mathcal{V}_1(G)$ consisting of the point valuations δ_x over X, and the valuation δ_∞, assigning 1 to every nonempty open set $U \subseteq G$.

Lemma B.1. *Every valuation $\nu \in \mathcal{V}_1(G)$ is of the form $\sum_{x\in G} \lambda_x\delta_x + \lambda_\infty\delta_\infty$, such that $\sum_{x\in G} \lambda_x + \lambda_\infty = 1$.*

Proof. For the valuation $\nu : \mathcal{O}(G) \to \mathbb{I}^{\uparrow}$, we define for $x \in X$, $\lambda_x = \nu(X \setminus \{x\})$, and $\lambda_\infty = 1 - \sum_{x\in X} \lambda_x$. Let $\nu' := \sum_{x\in G} \lambda_x\delta_x + \lambda_\infty\delta_\infty$. We claim $\nu = \nu'$.

First of all, observe that ν' is indeed a continuous valuation. Clearly, for all finite $F \subseteq X$, $\sum_{x\in F} \lambda_x = 1 - \nu(X \setminus F)$, hence $1 - \sum_{x\in X} \lambda_x$ exists and is indeed a nonnegative real number. For all finite $F \subseteq X$, define $\nu_F := \sum_{x\in F} \lambda_x\delta_x + \lambda_\infty\delta_\infty$. Then we have that $\nu_F(\emptyset) = 0$, $\nu_F(U) \leq \nu_F(V)$ whenever $U \subseteq V$, and $\nu_F(U \cup V) + \nu_F(U \cap V) = \nu_F(U) + \nu_F(V)$, hence all these hold for ν', as well. Moreover, $\nu'(X) = 1$, and ν' is obviously continuous, making it a continuous valuation.

Next we show by induction that $\nu(V_n) = \nu'(V_n)$ for all open sets of the form $V_n := \{(i, j) \in X |\ i \geq n\}$. For $V_0 = X$, this is clear. So suppose $\nu(V_n) = \sum_{x\in V_n} \lambda_x + \lambda_\infty$. For each $x \in V_n \setminus V_{n+1}$, $V_{n+1} \cup \{x\}$ is open, and we have

$$\nu(V_{n+1} \cup \{x\}) = \nu(X) + \nu(V_{n+1}) - \nu(X \setminus \{x\}) = \nu(V_{n+1}) + \lambda_x.$$

But then by continuity of ν,

$$\bigvee_{F\subseteq_{fin}V_n\setminus V_{n+1}} \nu(V_{n+1})+\sum_{x\in F}\lambda_x = \bigvee_{F\subseteq_{fin}V_n\setminus V_{n+1}} \nu(V_{n+1}\cup F) = \nu(V_n) = \sum_{x\in V_n}\lambda_x+\lambda_\infty,$$

hence

$$\nu(V_{n+1}) = \nu(V_n) - \sum_{x\in V_n\setminus V_{n+1}}\lambda_x = \sum_{x\in V_{n+1}}\lambda_x + \lambda_\infty,$$

by induction hypothesis.

Finally, we show the general case $\nu(U) = \nu'(U)$, similarly. Let $k = \min\{i \in \mathbb{N}|\ (i,j) \in U\}$, then for all $x \in V_k$, $U \cup \{x\}$ is open. Hence we can show as above,

$$\nu(U) = \nu(V_k) - \sum_{x\in V_n\setminus U}\lambda_x = \sum_{x\in U}\lambda_x + \lambda_\infty,$$

showing $\nu = \nu' = \sum_{x\in G}\lambda_x\delta_x + \lambda_\infty\delta_\infty$. □

Let X be any topological space, and $\mathcal{B}(X)$ denote the set of Borel sets of X, i.e. the smallest σ-algebra containing $\mathcal{O}(X)$. Recall that a continuous Borel probability measure over X is a map $\mu : \mathcal{B}(X) \to \mathbb{I}^\uparrow$, such that for all A, B and increasing chains $\{A_i\}_{i\in\mathbb{N}}$ in $\mathcal{B}(X)$, $\mu(\emptyset) = 0$, $\mu(X) = 1$, $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$, $A \subseteq B \Rightarrow \mu(A) \le \mu(B)$, and $\mu(\bigcup_{i\in\mathbb{N}} A_i) = \bigvee_{i\in\mathbb{N}} \mu(A_i)$. In the scope of the above lemma, we conjecture the following.

Lemma B.2. *The set $W := \{\nu \in \mathcal{V}_1(G)|\, \nu = \sum_{x\in G}\lambda_x\delta_x+\lambda_\infty\delta_\infty$ and $\lambda_\infty > 0\}$ is not open in the weak topology of $\mathcal{V}_1(G)$.*

Proof. Recall that the weak topology on $\mathcal{V}_1(G)$ has a subbasis given by sets of the form $\langle U, r\rangle := \{\nu \in \mathcal{V}_1(G)|\ \nu(U) > r\}$. We show that W does not contain any finite intersection of such subbasic open sets. Let $\bigcap_{i\in F}\langle U_i, r_i\rangle$ be such a finite intersection such that all the U_i are nonempty. Then $\bigcap_{i\in F} U_i$ is nonempty, say $x \in \bigcap_{i\in F} U_i$. Thus δ_x is a continuous valuation and $\delta_x \in \bigcap_{i\in F}\langle U_i, r_i\rangle$. However, $\lambda_\infty = 0$ for the valuation δ_x, hence $\bigcap_{i\in F}\langle U_i, r_i\rangle$ is not a subset of W. □

Lemma B.3. *Let $(\nu_i)_{i\in\mathbb{N}}$ be a sequence of valuations over G converging to ν_∞ in the weak topology, such that $\nu_i = \sum_{x\in G}\lambda^i_x\delta_x + \lambda^i_\infty\delta_\infty$ and $\nu_\infty = \sum_{x\in G}\lambda^\infty_x\delta_x + \lambda^\infty_\infty\delta_\infty$. Then for all finite $F \subseteq G$, $\{1 - \sum_{x\in F}\lambda^i_x\}$ converges to $1 - \sum_{x\in F}\lambda^\infty_x$ in $\mathbb{I}^\uparrow$. Equivalently, for all finite $F \subseteq G$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $i \ge N$, $\sum_{x\in F}\lambda^i_x \le \sum_{x\in F}\lambda^\infty_x + \epsilon$.*

Proof. As $\nu_i \to \nu_\infty$, we have that for all finite $F \subseteq G$, $\nu_i(G\setminus F) \to \nu_\infty(G\setminus F)$ in $\mathbb{I}^\uparrow$. Thus $\{1 - \sum_{x\in F}\lambda^i_x\}$ converges to $1 - \sum_{x\in F}\lambda^\infty_x$, as claimed. □

Lemma B.4. *The set* $B := \{\nu \in \mathcal{V}_1(G) | \, \nu = \sum_{x\in G} \lambda_x \delta_x + \lambda_\infty \delta_\infty$ and $\lambda_\infty = 0\}$ *is sequentially closed in the weak topology of* $\mathcal{V}_1(G)$.

Proof. We show that if $(\nu_i)_{i\in\mathbb{N}}$ is a sequence in B converging to some $\nu_\infty \in \mathcal{V}_1(X)$ in the weak topology, then $\nu_\infty \in B$. Let $\nu_i = \sum_{x\in G} \lambda^i_x \delta_x$ and $\nu_\infty = \sum_{x\in G} \lambda^\infty_x \delta_x + \lambda^\infty_\infty \delta_\infty$, then we have to show $\lambda^\infty_\infty = 0$.

Assume for a contradiction $\lambda^\infty_\infty = r > 0$. Let now $\frac{r}{3} > \epsilon > 0$, and choose $F \subseteq X$ finite such that $\sum_{x\in G\setminus F} \lambda^\infty_x \leq \epsilon$. Define $V_n := \{(k,l) \in X | \, k \geq n\} \setminus F$. As $(\nu_i)_{i\in\mathbb{N}}$ converges to ν_∞, we can w.l.g. assume that for all $i \in \mathbb{N}$, $\nu_i(V_i) \geq \nu_\infty(V_i) - \epsilon > \frac{2r}{3}$. Thus for each $i \in \mathbb{N}$, we find a finite $F_i \subseteq V_i$ such that $\sum_{x\in F_i} \lambda^i_x > \frac{r}{2}$. Let $K(F_i) := min\{k | \, (k,l) \in F_i\}$, then $\{K(F_i)\}_{i\in\mathbb{N}}$ is an unbounded sequence, and again w.l.g we can assume it is strictly increasing. For each $i \in \mathbb{N}$, let $L(F_i) := max\{l | \, (k,l) \in F_h \text{ for some } h \leq i\}$. Furthermore we define $A_0 := \{(k,l) \in G | \, k \leq K(F_1) \text{ and } l \leq L(F_1)\}$, and inductively

$$A_n := \{(k,l) \in G | \, K(F_n) < k \leq K(F_{n+1}) \text{ and } l \leq L(F_{n+1})\}.$$

Then we get that $\bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Moreover, it is not hard to see that $G \setminus \bigcup_{n\in\mathbb{N}} A_n$ is open in G; the reason is that for each fixed $k_0 \in \mathbb{N}$, $\{l | \, (k_0, l) \in \bigcup_{n\in\mathbb{N}} A_n)\}$ is bounded. Moreover the set $V := (G \setminus \bigcup_{n\in\mathbb{N}} A_n) \cup F$ is open and disjoint with all the F_i, as F is disjoint with all the F_i. Now we have $\nu_i(V) < 1 - \frac{r}{2}$, because $\sum_{x\in F_i} \lambda^i_x > \frac{r}{2}$. However, we also have

$$\nu_\infty(V) = r + \sum_{x\in V} \lambda^\infty_x \geq r + \sum_{x\in F} \lambda^\infty_x > 1 - \frac{r}{3}.$$

This clearly contradicts that $(\nu_i)_{i\in\mathbb{N}}$ converges to ν_∞. $\square$

Corollary B.5. *The space* $\mathcal{V}_1(G)$ *equipped with the weak topology is not sequential, hence not a qcb-space.*

Proof. By Lemma B.2, the set W is not open in the weak topology of $\mathcal{V}_1(G)$, but by Lemma B.4, it is sequentially open, as it is the complement of B. Thus $\mathcal{V}_1(G)$ with the weak topology is not sequential. $\square$